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Learning to Learn

The Art of Doing Science and Engineering

Session 23: Mathematics

Topic Outline

- What is Mathematics?
- Five Schools of Mathematical Thought
- The “Real Meaning” of Mathematics
- Languages Revisited
- A New Mathematics
- Final Considerations

What is Mathematics?

Like air, water and language, "Mathematics is in the background" and often taken for granted.

Nevertheless it plays a central role in science and engineering.

"Mathematics is what is done by Mathematicians, and Mathematicians are those who do Mathematics"

"Mathematics is the language of clear thinking"

… it's like Languages

There are many natural languages, but essentially only one language of Math

- Although an artificial "made-up" language, Mathematics is *universally* accepted (and possibly a better "language" than most languages)
- The Romans wrote VII, the Arabic notation is 7, and the binary notation is 111, but they all represent the *same idea* … 7 is always a 7
- Witness our own legal and tax codes to see just how inadequate the English language is for clear thinking and representation

Further defining Mathematics

Five schools of thought have described the nature of Mathematics (none satisfactorily)

- Platonic
- Formalists
- Logical
- Intuitionists
- Constructivists

Often grouped together

Platonic School

Basic tenet: ideas are more real than the physical world

- Plato claimed the idea of a chair was more real than any particular chair
- Humans infer things, e.g., a 2D eyeball seeing a 3D world
- All the world’s theorems were/are already in existence, just waiting to be discovered. They were not created
- but… Platonic school of thought doesn’t account for changing definitions in Mathematics, and how they evolved. Where were all those theorems waiting?

Formalists School

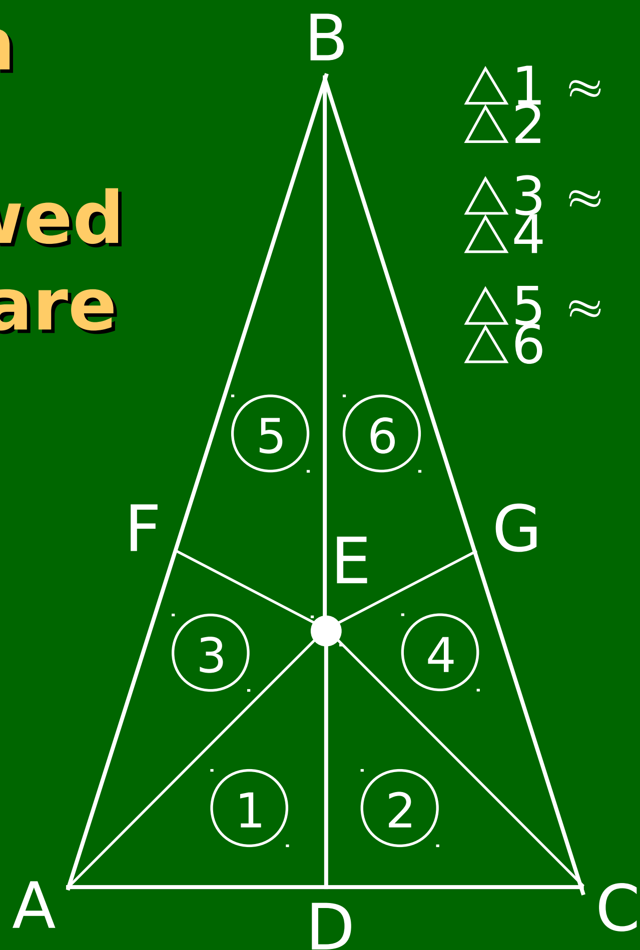
Basic tenet: mathematics was developed as a strictly mechanical process

- Math is simply manipulation of abstract strings of symbols, with no inherent meaning in themselves
- Hilbert, a popular Formalist said, “when rigor enters, meaning departs” – pay no attention to meaning!
- This school very popular among AI experts
- But… with no meaning, how is Math useful? How might we have predicted the locations of unknown planets, atomic bomb results, space flight, etc.?

A Formalist Proof

A well-known Middle Age “proof” showed all triangles are isosceles

- Bisect ∠B, and make the ⊥ bisector of line AC (at point D).
- From where these lines meet (at point E), work around to make triangles of equal angle



△1 ≈ △2

△3 ≈ △4

△5 ≈ △6

} ∴ **AB=BC**

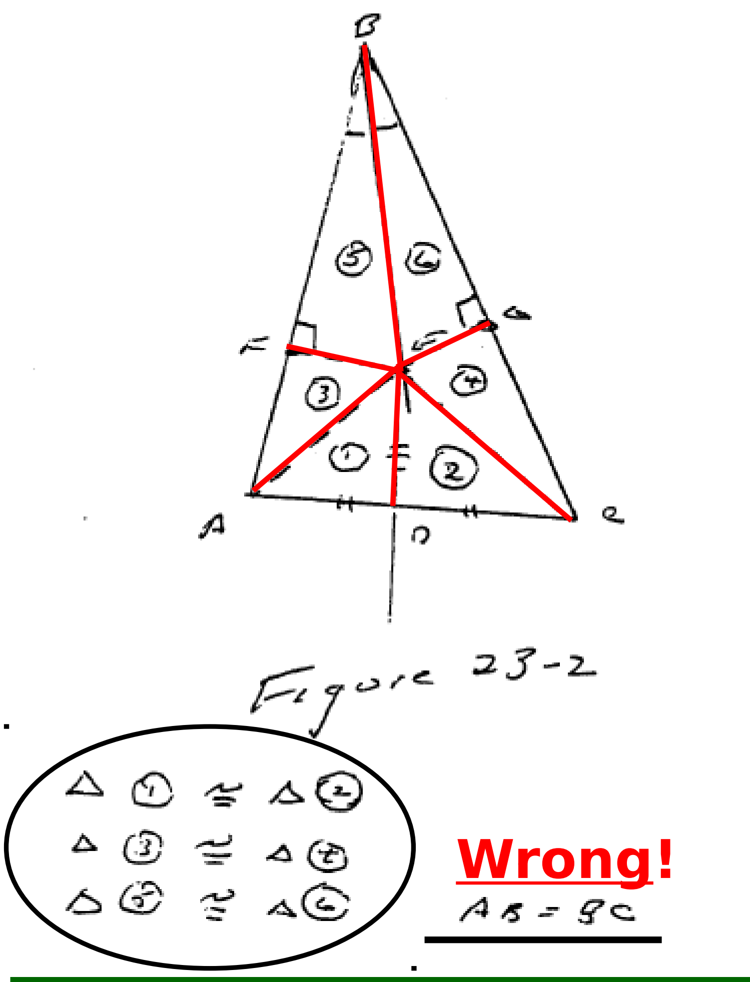
…Wrong!

Clearly, this theorem proof is wrong, but it followed an accepted style of thinking

(original drawing, annotated)

Bisect ∠B, and make the ⊥ bisector of line AC (at point D).

From where these lines meet (at point E), work around to make triangles of equal angle and length



Logical School

Basic tenet: all Mathematics is merely logic, and not necessarily truth

- Based on the principles espoused in the huge 3-volume Russell & Whitehead books, largely abandoned in recent times
- “Pure mathematics consists entirely of assertions to the effect that, if such and such a proposition it true of anything, then such and such another proposition is true of that thing”
- but… Logical school doesn’t account for the unreasonable effectiveness of Mathematics

Logical School

Hamming illustrated the counter-example of Cauchy’s Theorem

- If a student brought him a proof that Cauchy’s Theorem was false, i.e., could not be derived from the usual assertions, he’d be interested but in the long run he knows Cauchy’s Theorem is true.
- Mathematics does not exclusively follow from the assumptions, but rather the assumptions often follow from the theorems we “believe to be true”

Intuitionists School

Basic tenet: to use Math in the real world, you must have an intuition about it

- Intuitionists essentially ignore rigor. They say there is a valid ground between “yes” and “no”
- No presently proved theorem is really “proved”, rather the future will patch up earlier results… meaning we don’t ever fully prove anything!
- but… we must admit to a changing standard of rigor, meaning some proofs are just more convincing than others, and (perhaps) none likely reach total certainty

Constructivists School

Basic tenet: you must give explicit methods of constructing anything in Math

- Constructionists don’t rely on the accepted postulates, but say “I’ll believe something exists when I’m shown how to build it”
- Many in Computer Science would gravitate toward this school (though they probably don’t know it)
- but… this school is too strict and excludes too much of what we find valuable in practical Mathematics

The Five Schools of Mathematics

- None of the five schools of Mathematics have proved to be generally popular or accepted
- Hamming admits that he tends to belong to two of them (Intuitionist & Constructionist), although none is completely defensible
- None by itself can account for what we do in Mathematics, e.g. design and build a rocket, then take it to the moon

The “Real Meaning” of Math

The match between computing and the real world is not as good as we would like

- It would be simple to say the only real numbers are the bit patterns a machine generates, and that a Mathematician's “real numbers” are fictitious
- but... meanings change – the numbers in a machine suffer from truncation and round-off error, making them less “real”

Languages Revisited

We tend to identify words (names of things) with the object

- Lewis Carroll, a Logician, got into meta-linguistics when he distinguished between an object, the object's name, and the name of the name of the object – which of these represents "the object"?
- Meanings come from how things are manipulated, not how the words are said, e.g. Plato's chair
- but… would your son "Charles" be the same person if he had been named "Willy"?

Languages Revisited

How can we define a Language?

- Any dictionary must be circular – the first word you look up is defined by some other words
- If you point at a horse and say “horse”, do you mean the horse, its color, its name, all mammals, etc.?
- Also, peoples’ different *beliefs* create different meanings for the same words
- The meanings of words must be “described” rather than “prescribed” – meanings weren’t any more predetermined than Mathematical postulates were

A New Mathematics

Often we have to create new definitions as Mathematics evolves to new situations

- In creating (discovering?) Error Correction Codes, Hamming had to redefine 1+1 as equaling 0, not 2
- Gödel's Theorem, which stated that any proof cannot be self-consistent – it’s impossible to prove a system only within the context of the system – is really a theory about discrete symbols, not simply Mathematics

More on New Mathematics

Proven Mathematical Models won't solve everything for us

- Language systems, and Mathematics, each fall within the domain of Gödel. There are a lot of things we cannot do within the system of a computer (e.g. the Halting Problem).
- Our predecessors did the easy problems, we are doing the harder problems, and our successors will have to tackle the hardest ones.

Final Considerations

Mathematics will not always fit well into every field or problem

The Math that got us to the moon won't get us to Mars – it will require new Math

"Everything really worth knowing cannot be easily stated"